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Displacement Discontinuity over a Transversely Isotropic Elastic Half-Space

Abstract: The paper presents a solution to the elasticity problem where the discontinuity is in the displacement component parallel to the plane area inside the transversely isotropic medium. The work previously performed on discontinuity problems is also discussed.

Introduction

A close relationship exists between the theory of dislocation and the displacement discontinuity problem in the linear theory of elasticity. A displacement discontinuity can be constructed by a sequence of imaginary cutting, straining, and welding operations. For example consider a surface S, inside an elastic body, over which an imaginary cut is made. Give the two faces of the imaginary cut a relative displacement equal to a vector $\mathbf{B}_i(x, y, z)$ defined over S. Material may be added or removed to preserve material continuity across S. The entire imaginary cut is then welded together. There is an internal stress field caused by this displacement discontinuity, which is wholly characterized by the vector $\mathbf{B}_i(x, y, z)$ defined over the surface S. One may look at this problem from the point of view of the theory of continuous distributions of dislocations. Instead of the imaginary cutting, straining, and welding operation, we consider a distribution of elementary dislocation loops over the surface S. Imagine a Burgers circuit passing through S at a point (x, y, z). The Burgers vector at the point (x, y, z) due to all the dislocation lines threading the circuit is identical to the displacement discontinuity described earlier.

Dislocation is in fact a particular kind of line defect. A distribution of dislocation loops over a closed plane surface S represents a type of surface defect such as a thin crack in a crystalline solid or slippage between crystals. The elastic continuum analogue of this type of surface defect is a displacement discontinuity over a plane bounded area. It is not expected that the elasticity theory can give a detailed description of the behavior of cracks or lattice defects. An immediate result obtained from the continuum idealization is in the energy changes associated with the presence of defects. The knowledge of the strain energy may have some application in crack propagation and in the change of orientation of crystal boundaries. We shall assume that the elastic body is homogeneous and transversely isotropic. The general discontinuity problem can be constructed from two cases: (a) where the discontinuity vector is perpendicular to the surface S, and (b) where this vector is tangent to the surface S. The former case has been solved by Berry and Sales;[1] and the present paper, presenting the solution to the tangential discontinuity case, completes the general problem.

Berry and Sales have applied their solution to a physical situation of interest in mining and geology. They use the idealized model of the earth as a homogeneous, semi-infinite, transversely isotropic, elastic medium and apply the idea of a displacement discontinuity to the study of subsidence in a mining excavation at some depth below the earth surface. They find close correspondence between theoretical predictions and actual measurements obtained from some British coal fields. It is pointed out that although the displacement itself may be large (approximately four feet) compared to the usual elongation measurement in testing laboratories, as long as the strain is classically small, linear elasticity theory applies. Such is the case outside the immediate neighborhood of the excavation.

The stress field due to an arbitrary displacement discontinuity over a bounded plane area in a semi-infinite or infinite solid, has been determined by Rongved and Frasier[2], and Rongved[3], when the elastic medium is isotropic. As we have mentioned earlier, Berry and Sales solved this problem for a transversely isotropic elastic medium when the discontinuity was in the displacement component normal to the plane area. Their solutions were in terms of integrals over the area of discontinuity. Also, a special case of constant discontinuity over a rectangular area was worked out in closed form. In this paper, the elasticity problem, where the discontinuity is in the displacement component parallel to the plane area inside the transversely isotropic medium, is studied

The solution is obtained in the form of stress functions analogous to those used for isotropic materials. As in the isotropic case, these stress functions are expressed in terms of integrals over the area of discontinuity. First, the problem is solved for an infinite medium with the displacement discontinuity. From the conditions of continuity of certain stress and displacement components across the plane containing the discontinuity, it is possible to reduce the general solution to a single "harmonic" function. Since the value of this function is known in this plane, it may be determined by means of Green's formula.

The case where the discontinuity lies in a semi-infinite medium is obtained by applying the method of images and by introducing a residual solution which gives the desired traction-free plane surface. A special case of constant displacement over a rectangular plane area is presented in closed form. If the discontinuity occurs over a circular area, the use of integrals of Bessel functions simplifies the analysis considerably. The result is presented in the Appendix.

The analysis in this paper is based on Rongved's solution to the isotropic elastic problem (2), and the potential functions method first introduced by Elliott[4,5].*

Finally, it is pertinent to point out that steady state dynamic elasticity problems with moving boundary conditions, moving at a uniform velocity less than the propagation velocities of the material, have the same forms of potential functions as the corresponding static problems. Therefore, a knowledge of the solution to a static problem will lead to the solution to the dynamic problem through a slight change in the analysis. The two dimensional analog to this method has been discussed by Stroth[7], who treated material with more general anisotropy. This method and its restrictions will be discussed in a forthcoming note. It follows, that by the use of this technique, the present infinite solid solution is also valid if the discontinuity is moving at a constant velocity along the z-axis.

Potential functions

In the linear elasticity theory, the fundamental system of field equations are the linearized strain-displacement equations, the linear stress-strain relations (Hooke's Law), and the stress equations of equilibrium. We shall restrict ourselves to transversely isotropic, homogeneous media with no body forces. The symmetry axis of the material is taken to be the z-axis. The strain components are defined as

$$\begin{aligned} e_{xx} &= \frac{\partial u}{\partial x}, \qquad e_{yy} = \frac{\partial v}{\partial y}, \qquad e_{zz} = \frac{\partial w}{\partial z} \\ e_{yz} &= \left(\frac{\partial v}{dz} + \frac{\partial w}{\partial y}\right), \qquad e_{zx} = \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) \\ e_{xy} &= \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right), \end{aligned} \tag{1a}$$

where u, v, w are the displacement components in the x, y, z direction.

The equilibrium equations are

$$\begin{aligned} \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} &= 0 \\ \frac{\partial \sigma_{yx}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{yz}}{\partial z} &= 0 \\ \frac{\partial \sigma_{zx}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} &= 0. \end{aligned} \tag{1b}$$

And finally, the stress-strain relations are

$$\begin{aligned} \sigma_{xx} &= c_{11}e_{xx} + c_{12}e_{yy} + c_{13}e_{zz} \\ \sigma_{yy} &= c_{12}e_{xx} + c_{11}e_{yy} + c_{13}e_{zz} \\ \sigma_{zz} &= c_{13}(e_{xx} + e_{yy}) + c_{33}e_{zz} \\ \sigma_{yz} &= c_{44}e_{yz} \qquad \sigma_{zx} = c_{44}e_{zx} \\ \sigma_{xy} &= \tfrac{1}{2}(c_{11} - c_{12})e_{xy}, \end{aligned} \tag{1c}$$

where the c_{ij} are the elastic constants.

It is known[4,8] that Eqs. (1a, b, c) are satisfied if the displacements and stresses are expressed in terms of three "harmonic" functions ϕ_1, ϕ_2, ψ which are solutions of

$$\begin{aligned} \left(\nabla_1^2 + \frac{\partial^2}{\partial z_j^2}\right)\phi_j &= 0 \qquad (j = 1, 2) \\ \left(\nabla_1^2 + \frac{\partial^2}{\partial z_3}\right)\psi &= 0, \end{aligned} \tag{2}$$

where

$$z_i = z/\sqrt{\nu_i} \qquad (i = 1, 2, 3),$$

* References 1 and 6 present further information on work in this field.

$\nabla_1^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$, ν_1 and ν_2 are roots of the equation

$$c_{11}c_{44}\nu^2 + [c_{13}(2c_{44} + c_{13}) - c_{11}c_{13}]\nu + c_{33}c_{44} = 0, \quad (3)$$

and

$$\nu_3 = \frac{2c_{44}}{c_{11} - c_{12}}. \quad (4)$$

The displacement components in Cartesian coordinates are

$$\begin{aligned} u &= \frac{\partial \phi_1}{\partial x} + \frac{\partial \phi_2}{\partial x} + \frac{\partial \psi}{\partial y} \\ v &= \frac{\partial \phi_1}{\partial y} + \frac{\partial \phi_2}{\partial y} - \frac{\partial \psi}{\partial x} \\ w &= \frac{k_1}{\sqrt{\nu_1}} \frac{\partial \phi_1}{\partial z_1} + \frac{k_2}{\sqrt{\nu_2}} \frac{\partial \phi_2}{\partial z_2}, \end{aligned} \quad (5)$$

where k_1, k_2 are given by

$$\nu = \frac{kc_{33}}{kc_{44} + (c_{13} + c_{44})} = \frac{k(c_{13} + c_{44}) + c_{44}}{c_{11}}. \quad (6)$$

From the above relations, three stress components of immediate interest are

$$\begin{aligned} \frac{\sigma_{zz}}{c_{44}} &= (1 + k_1) \frac{\partial^2 \phi_1}{\partial z_1^2} + (1 + k_2) \frac{\partial^2 \phi_2}{\partial z_2^2} \\ \frac{\sigma_{xz}}{c_{44}} &= \frac{(1 + k_1)}{\sqrt{\nu_1}} \frac{\partial^2 \phi_1}{\partial x \partial z_1} + \frac{(1 + k_2)}{\sqrt{\nu_2}} \frac{\partial^2 \phi_2}{\partial x \partial z_2} + \frac{1}{\sqrt{\nu_3}} \frac{\partial^2 \psi}{\partial y \partial z_3} \\ \frac{\sigma_{yz}}{c_{44}} &= \frac{(1 + k_1)}{\sqrt{\nu_1}} \frac{\partial^2 \phi_1}{\partial x \partial z_1} + \frac{(1 + k_2)}{\sqrt{\nu_2}} \frac{\partial^2 \phi_2}{\partial x \partial z_2} - \frac{1}{\sqrt{\nu_3}} \frac{\partial^2 \psi}{\partial y \partial z_3}. \end{aligned} \quad (7)$$

The roots ν_1, ν_2 may be real positive or complex conjugates. The root ν_3 is always real and positive. It is specified that in the case of complex roots, $\sqrt{\nu_1}$ and $\sqrt{\nu_2}$ are to have positive real parts.

Infinite solid

If the discontinuity in displacement occurs over a plane bounded area A, which is thought of as coinciding with the $z = 0$ plane, then the general problem may be stated in terms of the two following conditions:

(a) The stress components σ_{zz}, σ_{xz}, σ_{yz} are continuous everywhere across the plane $z = 0$. Displacements (u, v, w) are continuous across $z = 0$ except over A, where the discontinuity is prescribed.
(b) Stresses and displacement must vanish at infinity.

The general case can be obtained by superposition of three different problems in which the discontinuity occurs only in u, v, and w, respectively. The case where discontinuity exists only in the component w has been solved by Berry and Sales.[1] It is clear that the analyses for the problems where there is only a u component discontinuity or only a v component discontinuity are essentially the same. We shall therefore discuss the elasticity problem of an infinite solid with a displacement discontinuity in the component along the x direction over an area A on the $z = 0$ plane.

Let plain and the prime quantities be associated with half-spaces defined by $z \geq 0$, and $z \leq 0$, respectively. Our boundary conditions are such that at $z = 0$,

$$\sigma_{xz} = \sigma'_{xz} \quad (8)$$

$$\sigma_{zy} = \sigma'_{zy} \quad (9)$$

$$\sigma_{zz} = \sigma'_{zz} \quad (10)$$

$$\bar{u} + u = u' \quad (11)$$

$$v = v' \quad (12)$$

$$w = w', \quad (13)$$

where $\bar{u}(x, y)$ is the prescribed discontinuity over A, and is defined to be zero everywhere outside A.

We are now concerned with two half spaces $z \geq 0$, $z \leq 0$ to which are assigned the potential functions (ϕ_1, ϕ_2, ψ) and $(\phi'_1, \phi'_2, \psi')$ respectively. The boundary conditions to be satisfied at $z = 0$ are Eqs. (8) to (13). Examine Eqs. (8) to (13) in terms of their potential function representations in Eqs. (5) and (7).

Let

$$\phi_j(x, y, z_j) = -\phi'_j(x, y, -z_j) \qquad (j = 1, 2) \quad (14)$$

$$\psi(x, y, z_3) = -\psi'(x, y, -z_3). \quad (15)$$

Then Eqs. (8), (9), and (13) are satisfied.

Let

$$\phi_j = \frac{(-1)^{1+j}}{(1 + k_j)} H(x, y, z_j) \qquad (j = 1, 2) \quad (16)$$

where $\nabla^2 H(x, y, z) = 0$.

This satisfies Eq. (10).

Let

$$\frac{\partial \psi}{\partial x}(x, y, z) = \frac{1}{1 + k_1} \frac{\partial H(x, y, z)}{\partial y} - \frac{1}{1 + k_2} \frac{\partial H(x, y, z)}{\partial y}. \quad (17)$$

This satisfies Eq. (12). In Eqs. (14) to (17), we have made some intuitive guesses to the displacement field without violating the boundary conditions.

Finally to satisfy Eq. (11), we write

$$2\left\{\frac{1}{1+k_1}\frac{\partial H(x, y, z)}{\partial x} + \frac{1}{1+k_2}\frac{\partial H(x, y, z)}{\partial x} + \frac{\partial \psi(x, y, x)}{\partial y}\right\} = -D_x(x, y, z), \tag{18}$$

where we define D_x to be harmonic in $z \geq 0$, equal to $\bar{u}(x, y)$ on A at $z = 0$, and equal to zero over remaining portion of $z = 0$. It is clear that the problem is reduced to finding a harmonic function regular in $z > 0$ with its boundary value prescribed at $z = 0$, i.e., a Dirichlet problem. We can write

$$D_x(x, y, z) = -\frac{1}{2\pi}\frac{\partial}{\partial z}\int_A \frac{\bar{u}(\xi, \mu)}{r_0}\, d\xi\, d\mu, \tag{19}$$

where $r_0 = (x - \xi)^2 + (y - \mu)^2 + z^2$.

Since $H(x, y, z)$ and $\psi(x, y, z)$ and D_x are all harmonic we have from Eqs. (17), (18), and (16) that

$$\frac{\partial H}{\partial z}(x, y, z) = +\frac{(1+k_1)(1+k_2)}{4\pi(k_1 - k_2)}\frac{\partial}{\partial x}\int_A \frac{\bar{u}(\xi, \mu)}{r_0}\, d\xi\, d\mu \tag{20}$$

i.e.,

$$H(x, y, z) = +\frac{(1+k_1)(1+k_2)}{4\pi(k_1 - k_2)} \times \int_A \frac{\bar{u}(\xi, \mu)(x - \xi)}{r_0(r_0 + z)}\, d\xi\, d\mu \tag{21}$$

$$\psi(x, y, z) = \frac{1}{4\pi}\int_A \frac{u(\xi, \mu)(y - \mu)}{r_0(r_0 + z)}\, d\xi\, d\mu. \tag{22}$$

Equations (5), (14), (15), (16), (21), and (22) form a complete solution to the elasticity problem.* Stresses and displacements are expressed in terms of partial derivatives of potential functions. These functions are known apart from arbitrary functions $zf(x, y) + g(x, y)$. They are usually eliminated by the conditions of stress and displacement at infinity.

It may be remarked that instead of Eqs. (8) to (13) it is possible to state the boundary condition more explicitly (as by Berry and Sales,[1] Eq. 8). We can consider the half-space $z \geq 0$, such that, on $z = 0$

$$\begin{aligned} v &= 0 \\ \sigma_{zz} &= 0 \\ u &= -\tfrac{1}{2}\bar{u} \quad \text{on } A \\ u &= 0 \qquad \text{elsewhere on } z = 0. \end{aligned} \tag{23}$$

However, it is felt that Eqs. (8) to (13) are a more fundamental statement of the boundary conditions of a discontinuity problem. The displacement, and some of the stress components may be expressed in terms of integrals over the discontinuity. We write

$$I_i = \int_A \frac{(x - \xi)}{r_{0i}(r_{0i} + z_i)}\bar{u}(\xi, \mu)\, d\xi\, d\mu \tag{24}$$

$$J_i = \int_A \frac{(y - \mu)}{r_{0i}(r_{0i} + z_i)}\bar{u}(\xi, \mu)\, d\xi\, d\mu. \tag{25}$$

Then, the displacement components are

$$\begin{aligned} 4\pi w &= -\frac{k_1(1+k_2)}{\sqrt{\nu_1}(k_2 - k_1)}\frac{\partial I_1}{\partial z_1} + \frac{k_2(1+k_1)}{\sqrt{\nu_2}(k_2 - k_1)}\frac{\partial I_2}{\partial z_2} \\ 4\pi v &= -\frac{1+k_2}{(k_2 - k_1)}\frac{\partial I_1}{\partial y} + \frac{1+k_1}{(k_2 - k_1)}\frac{\partial I_2}{\partial y} + \frac{\partial J_3}{\partial x} \\ 4\pi u &= -\frac{1+k_2}{(k_2 - k_1)}\frac{\partial I_1}{\partial x} + \frac{1+k_1}{(k_2 - k_1)}\frac{\partial I_2}{\partial x} - \frac{\partial J_3}{\partial y}. \end{aligned} \tag{26}$$

$$\begin{aligned} \frac{4\pi}{c_{44}}\sigma_{zz} &= -\frac{(1+k_1)(1+k_2)}{k_2 - k_1}\left[\frac{\partial^2 I_1}{\partial z_1^2} - \frac{\partial^2 I_2}{\partial z_2^2}\right] \\ \frac{4\pi}{c_{44}}\sigma_{xz} &= -\frac{(1+k_1)(1+k_2)}{k_2 - k_1}\left[\frac{1}{\sqrt{\nu_1}}\frac{\partial^2 I_1}{\partial x \partial z_1} - \frac{1}{\sqrt{\nu_2}}\frac{\partial^2 I_2}{\partial x \partial z_2}\right] - \frac{1}{\sqrt{\nu_3}}\frac{\partial^2 J_3}{\partial y \partial z_3} \\ \frac{4\pi}{c_{44}}\sigma_{yz} &= -\frac{(1+k_1)(1+k_2)}{k_2 - k_1}\left[\frac{1}{\sqrt{\nu_1}}\frac{\partial^2 I_1}{\partial y \partial z_1} - \frac{1}{\sqrt{\nu_2}}\frac{\partial^2 I_2}{\partial y \partial z_2}\right] + \frac{1}{\sqrt{\nu_3}}\frac{\partial^2 J_3}{\partial x \partial z_3}. \end{aligned} \tag{27}$$

Note 1: The stresses and displacements are all real even if ν_1, ν_2, and k_1, k_2 are pairs of complex conjugates.

Note 2: To continue the solution into the $z < 0$ space, it may be sufficient to state that, from inspection of Eqs. (14) and (15), u and v are odd functions of z, while w is an even function of z; and in terms of stress components, σ_{zz}, σ_{xx}, σ_{yy}, and σ_{xy} are odd functions of z, while σ_{xz}, σ_{yz} are even functions of z.

Note 3: In the studies of crack propagation and of dislocation theory, it is often useful to know the elastic strain energy due to the displacement discontinuity. This strain energy, sometimes called the self-energy, is given by

$$E_s = \frac{1}{2}\int_v \sigma_{mn}e_{mn}\, dv,$$

summed over m and n, and integrated over the entire medium. Through the use of equations (1a, b, c) and the divergence theorem, it can be shown that

$$E_s = \frac{1}{2}\int_A (\bar{u}\sigma_{xz})|_{z=0^+}\, dx\, dy,$$

* If A represents a circular area, the use of integrals of Bessel functions gives much simpler results. They are presented in the Appendix.

where A is the area over which the displacement discontinuity occurs.

Examples

If $\bar{u}(x, y) = 1$, and A is a rectangle defined by

$-a \leq x \leq a, \quad -b \leq y \leq b$, then

$$\frac{\partial I_i}{\partial z_i} = \log \frac{[r_{1i} - y + b][r_{3i} - y - b]}{[r_{2i} - y + b][r_{4i} - y - b]} \tag{28}$$

$$\frac{\partial J_i}{\partial x} = \frac{\partial I_i}{\partial y} = \log \frac{(r_{1i} + z_i)(r_{3i} + z_i)}{(r_{2i} + z_i)(r_{4i} + z_i)} \tag{29}$$

$$\begin{aligned}\frac{1}{2}\frac{\partial I_i}{\partial x} &= \tan^{-1} \left\{ \frac{(a - x) \tan \frac{1}{2}\left(\tan^{-1} \frac{b - y}{[(a - x)^2 + z_i^2]^{1/2}}\right)}{[(a - x)^2 + z_i^2]^{1/2} + z_i} \right\} \\ &+ \tan^{-1} \left\{ \frac{(a + x) \tan \frac{1}{2}\left(\tan^{-1} \frac{b + y}{[(x + a)^2 + z_i^2]^{1/2}}\right)}{[(a + x)^2 + z_i^2]^{1/2} + z_i} \right\} \\ &+ \tan^{-1} \left\{ \frac{(a + x) \tan \frac{1}{2}\left(\tan^{-1} \frac{b - y}{[(a + x)^2 + z_i^2]^{1/2}}\right)}{[(a + x)^2 + z_i^2]^{1/2} + z_i} \right\} \\ &+ \tan^{-1} \left\{ \frac{(a - x) \tan \frac{1}{2}\left(\tan^{-1} \frac{b + y}{[(a - x)^2 + z_i^2]^{1/2}}\right)}{[(a - x)^2 + z_i^2]^{1/2} + z_i} \right\}\end{aligned} \tag{30}$$

$$\begin{aligned}\frac{1}{2}\frac{\partial J_i}{\partial y} &= \tan^{-1} \left\{ \frac{(b - y) \tan \frac{1}{2}\left(\tan^{-1} \frac{a - x}{[(b - y)^2 + z_i^2]^{1/2}}\right)}{[(b - y)^2 + z_i^2]^{1/2} + z_i} \right\} \\ &+ \tan^{-1} \left\{ \frac{(b + y) \tan \frac{1}{2}\left(\tan^{-1} \frac{a + x}{[(b + y)^2 + z_i^2]^{1/2}}\right)}{[(b + y)^2 + z_i^2]^{1/2} + z_i} \right\} \\ &+ \tan^{-1} \left\{ \frac{(b + y) \tan \frac{1}{2}\left(\tan^{-1} \frac{a - x}{[(b + y)^2 + z_i^2]^{1/2}}\right)}{[(b + y)^2 + z_i^2]^{1/2} + z_i} \right\} \\ &+ \tan^{-1} \left\{ \frac{(b - y) \tan \frac{1}{2}\left(\tan^{-1} \frac{a + x}{[(b - y)^2 + z_i^2]^{1/2}}\right)}{[(b - y)^2 + z_i^2]^{1/2} + z_i} \right\}\end{aligned} \tag{31}$$

where

$$\begin{aligned} r_{1i}^2 &= (a - x)^2 + (b - y)^2 + z_i^2 \\ r_{2i}^2 &= (a + x)^2 + (b - y)^2 + z_i^2 \\ r_{3i}^2 &= (a + x)^2 + (b + y)^2 + z_i^2 \\ r_{4i}^2 &= (a - x)^2 + (b + y)^2 + z_i^2. \end{aligned} \tag{32}$$

The strain energy due to the dislocation has been found. It is

$$\begin{aligned} E_s &= \frac{c_{44}(\sqrt{\nu_1} - \sqrt{\nu_2})(1 + k_1)(1 + k_2)}{4\pi\sqrt{\nu_1\nu_2}(k_1 - k_2)} \cdot b \log \left[\frac{(a^2 + b^2)^{1/2} + b}{(a^2 + b^2)^{1/2} - b}\right] \\ &+ \frac{c_{44}}{4\pi\nu_3} a \log \left[\frac{(a^2 + b^2)^{1/2} + a}{(a^2 + b^2)^{1/2} - a}\right]. \end{aligned}$$

Semi-infinite solid

We consider a discontinuity over a region A on the plane $z = h$, inside the semi-infinite solid $z \geq 0$. Superpose two discontinuities of equal strength and opposite sign at planes $z = h$, and $z = -h$ inside a infinite solid. Then from Note 2 of the previous section, $\sigma_{xz} = \sigma_{yz} = 0$ on the plane $z = 0$. It is now necessary to find a residual solution which will not produce any singularity inside $z \geq 0$, be free of shear stress on $z = 0$, and will annihilate the normal stress at that plane. This is the same situation faced in some other half-space problems (Shield,[9] Eq. 4.2).

The stress that we need to annihilate is

$$\sigma_{zz} = -2c_{44}\left[\frac{\partial^2 H(x, y, h_1)}{\partial z_1^2} - \frac{\partial^2 H(x, y, h_2)}{\partial z_2^2}\right].$$

Define a residual potential in terms of function H (see Eq. 21).

$$\phi_i^R = +\frac{2\sqrt{\nu_i}}{(1 + k_i)}\frac{(-1)^{1+i}}{[\sqrt{\nu_1} - \sqrt{\nu_2}]} \times [H(x, y, z_i + h_1) - H(x, y, z_i + h_2)]. \tag{33}$$

It is found that on $z = 0$, its surface traction due to ϕ^R is

$$\sigma_{xz} = \sigma_{yz} = 0$$

and

$$\sigma_{zz} = +2c_{44}\left[\frac{\partial^2 H(x, y, h_1)}{\partial z_1^2} - \frac{\partial^2 H(x, y, h_2)}{\partial z_2^2}\right].$$

Also ϕ^R does not produce any singularity inside $z \geq 0$. The term ϕ^R therefore satisfies all the requirements of the residual potential.

If the infinite solid potential functions as defined in Eqs. (14), (15), (16), (21), and (22) are called $\phi_1^I(x, y, z)$, $\phi_2^I(x, y, z)$ and $\psi^I(x, y, z)$, respectively, then the potential functions for the semi-infinite solid can be written as

$$\begin{aligned} \phi_1 &= \phi_1^I(x, y, z - h) - \phi_1^I(x, y, z + h) + \phi_1^R \\ \phi_2 &= \phi_2^I(x, y, z - h) - \phi_2^I(x, y, z + h) + \phi_2^R \\ \psi &= \psi^I(x, y, z - h) - \psi^I(x, y, z + h). \end{aligned} \tag{34}$$

These potential functions are represented in terms of the integrals over the area A. Let

$$
\begin{aligned}
\bar{r}_{1ji}^2 &= (a - x)^2 + (b - y)^2 + (z_i - h_i)^2 \\
\bar{r}_{2ji}^2 &= (a + x)^2 + (b - y)^2 + (z_i - h_i)^2 \\
\bar{r}_{3ji}^2 &= (a + x)^2 + (b + y)^2 + (z_i - h_i)^2 \\
\bar{r}_{4ji}^2 &= (a - x)^2 + (b + y)^2 + (z_i - h_i)^2 \\
\bar{\bar{r}}_{1ji}^2 &= (a - x)^2 + (b - y)^2 + (z_i + h_i)^2 \\
\bar{\bar{r}}_{2ji}^2 &= (a + x)^2 + (b - y)^2 + (z_i + h_i)^2 \\
\bar{\bar{r}}_{3ji}^2 &= (a + x)^2 + (b + y)^2 + (z_i + h_i)^2 \\
\bar{\bar{r}}_{4ji}^2 &= (a - x)^2 + (b + y)^2 + (z_i + h_i)^2 \\
\bar{r}_{0ji}^2 &= (x - \xi)^2 + (y - \mu)^2 + (z_i - h_i)^2 \\
\bar{\bar{r}}_{0ji}^2 &= (x - \xi)^2 + (y - \mu)^2 + (z_i + h_i)^2,
\end{aligned}
\tag{35}
$$

and let

$$
\bar{I}_{ji} = \int_A \frac{(x - \xi)\bar{u}(\xi, \mu)\, d\xi\, d\mu}{\bar{r}_{0ji}(\bar{r}_{0ji} + z_i - h_i)} \tag{36}
$$

$$
\bar{\bar{I}}_{ji} = \int_A \frac{(x - \xi)\bar{u}(\xi, \mu)\, d\xi\, d\mu}{\bar{\bar{r}}_{0ji}(\bar{\bar{r}}_{0ji} + z_i + h_i)} \tag{37}
$$

$$
\bar{J}_{ji} = \int_A \frac{(y - \mu)\bar{u}(\xi, \mu)\, d\xi\, d\mu}{\bar{r}_{0ji}(\bar{r}_{0ji} + z_i - h_i)} \tag{38}
$$

$$
\bar{\bar{J}}_{ji} = \int_A \frac{(y - \mu)\bar{u}(\xi, \mu)\, d\xi\, d\mu}{\bar{\bar{r}}_{0ji}(\bar{\bar{r}}_{0ji} + z_i + h_i)}. \tag{39}
$$

The displacement may be expressed as (for $z > h$)

$$
\begin{aligned}
-4\pi w = {} & \frac{k_1(1 + k_2)}{\sqrt{\nu_1}(k_2 - k_1)} \left[\frac{\partial \bar{I}_{11}}{\partial z_1} - \frac{\partial \bar{\bar{I}}_{11}}{\partial z_1} \right] \\
& - \frac{2k_1(1 + k_2)}{(k_2 - k_1)[\nu_1^{1/2} - \nu_2^{1/2}]} \left[\frac{\partial \bar{I}_{11}}{\partial z_1} - \frac{\partial \bar{\bar{I}}_{12}}{\partial z_1} \right] \\
& + \frac{k_2(1 + k_1)}{\sqrt{\nu_2}(k_2 - k_1)} \left[\frac{\partial \bar{I}_{22}}{\partial z_2} - \frac{\partial \bar{\bar{I}}_{22}}{\partial z_2} \right] \\
& + \frac{2k_2(1 + k_1)}{(k_2 - k_1)[\nu_1^{1/2} - \nu_2^{1/2}]} \left[\frac{\partial \bar{I}_{21}}{\partial z_2} - \frac{\partial \bar{\bar{I}}_{22}}{\partial z_2} \right]
\end{aligned}
\tag{40}
$$

$$
\begin{aligned}
-4\pi u = {} & \frac{1 + k_2}{(k_2 - k_1)} \left[\frac{\partial \bar{I}_{11}}{\partial x} - \frac{\partial \bar{\bar{I}}_{11}}{\partial x} \right] \\
& - \frac{2\sqrt{\nu_1}(1 + k_2)}{(k_2 - k_1)[\nu_1^{1/2} - \nu_2^{1/2}]} \left[\frac{\partial \bar{I}_{11}}{\partial x} - \frac{\partial \bar{\bar{I}}_{12}}{\partial x} \right] \\
& - \frac{(1 + k_1)}{(k_2 - k_1)} \left[\frac{\partial \bar{I}_{22}}{\partial x} - \frac{\partial \bar{\bar{I}}_{22}}{\partial x} \right] \\
& + \frac{2\sqrt{\nu_2}(1 + k_1)}{(k_2 - k_1)[\nu_1^{1/2} - \nu_2^{1/2}]} \left[\frac{\partial \bar{I}_{21}}{\partial x} - \frac{\partial \bar{\bar{I}}_{22}}{\partial x} \right] \\
& + \left(\frac{\partial \bar{J}_{33}}{\partial y} - \frac{\partial \bar{\bar{J}}_{33}}{\partial y} \right)
\end{aligned}
\tag{41}
$$

$$
\begin{aligned}
-4\pi v = {} & \frac{(1 + k_2)}{(k_2 - k_1)} \left[\frac{\partial \bar{I}_{11}}{\partial y} - \frac{\partial \bar{\bar{I}}_{11}}{\partial y} \right] \\
& - \frac{2\sqrt{\nu_1}(1 + k_2)}{(k_2 - k_1)[\nu_1^{1/2} - \nu_2^{1/2}]} \left[\frac{\partial \bar{I}_{11}}{\partial y} - \frac{\partial \bar{\bar{I}}_{12}}{\partial y} \right] \\
& - \frac{(1 + k_1)}{(k_2 - k_1)} \left[\frac{\partial \bar{I}_{22}}{\partial y} - \frac{\partial \bar{\bar{I}}_{22}}{\partial y} \right] \\
& - \frac{2\sqrt{\nu_1}(1 + k_1)}{(k_2 - k_1)[\nu_1^{1/2} - \nu_2^{1/2}]} \left[\frac{\partial \bar{I}_{21}}{\partial y} - \frac{\partial \bar{\bar{I}}_{22}}{\partial y} \right] \\
& - \left(\frac{\partial \bar{J}_{33}}{\partial x} - \frac{\partial \bar{\bar{J}}_{33}}{\partial x} \right).
\end{aligned}
\tag{42}
$$

Example

If $u(\xi, \mu) = 1$, and A is a rectangular region with sides $2a$ and $2b$ parallel to the y and x axes respectively, then

$$
\frac{\partial \bar{I}_{ji}}{\partial z_i} = \log \frac{(\bar{r}_{1ji} - y + b)(\bar{r}_{3ji} - y - b)}{(\bar{r}_{2ji} - y - b)(\bar{r}_{4ji} - y - b)} \tag{43}
$$

$$
\frac{\partial \bar{I}_{ji}}{\partial y} = \log \frac{(\bar{r}_{1ji} + z_i - h_i)(\bar{r}_{3ji} + z_i - h_i)}{(\bar{r}_{2ji} + z_i - h_i)(\bar{r}_{4ji} + z_i - h_i)} \tag{44}
$$

$$
\begin{aligned}
\frac{1}{2}\frac{\partial \bar{I}_{ji}}{\partial x} = {} & \tan^{-1} \left\{ \frac{(a - x) \tan \frac{1}{2}\left(\tan^{-1} \frac{b - y}{[(a - x)^2 + (z_i - h_i)^2]^{1/2}} \right)}{[(a - x)^2 + (z_i - h_i)^2]^{1/2} + (z_i - h_i)} \right\} \\
& + \tan^{-1} \left\{ \frac{(a + x) \tan \frac{1}{2}\left(\tan^{-1} \frac{b + y}{[(a + x)^2 + (z_i - h_i)^2]^{1/2}} \right)}{[(a + x)^2 + (z_i - h_i)^2]^{1/2} + (z_i - h_i)} \right\} \\
& + \tan^{-1} \left\{ \frac{(a + x) \tan \frac{1}{2}\left(\tan^{-1} \frac{b - y}{[(a + x)^2 + (z_i + h_i)^2]^{1/2}} \right)}{[(a + x)^2 + (z_i - h_i)^2]^{1/2} + (z_i - h_i)} \right\} \\
& + \tan^{-1} \left\{ \frac{(a - x) \tan \frac{1}{2}\left(\tan^{-1} \frac{b + y}{[(a - x)^2 + (z_i - h_i)^2]^{1/2}} \right)}{[(a - x)^2 + (z_i - h_i)^2]^{1/2} + (z_i - h_i)} \right\}
\end{aligned}
\tag{45}
$$

$$\frac{1}{2}\frac{\partial \bar{J}_{ij}}{\partial y} = \tan^{-1}\left\{\frac{(b-y)\tan\frac{1}{2}\left(\tan^{-1}\frac{a-x}{[(b-y)^2+(z_i-h_i)^2]^{1/2}}\right)}{[(b-y)^2+(z_i-h_i)^2]^{1/2}+[z_i-h_i]}\right\}$$

$$+\tan^{-1}\left\{\frac{(b+y)\tan\frac{1}{2}\left(\tan^{-1}\frac{a+x}{[(b+y)^2+(z_i-h_i)^2]^{1/2}}\right)}{[(b+y)^2+(z_i-h_i)^2]^{1/2}+[z_i-h_i]}\right\}$$

$$+\tan^{-1}\left\{\frac{(b+y)\tan\frac{1}{2}\left(\tan^{-1}\frac{a-x}{[(b+y)^2+(z_i-h_i)^2]^{1/2}}\right)}{[(b+y)^2+(z_i-h_i)^2]^{1/2}+[z_i-h_i]}\right\}$$

$$+\tan^{-1}\left\{\frac{(b-y)\tan\frac{1}{2}\left(\tan^{-1}\frac{a+x}{[(b-y)^2+(z_i-h_i)^2]^{1/2}}\right)}{[(b-y)^2+(z_i-h_i)^2]^{1/2}+[z_i-h_i]}\right\}. \quad (46)$$

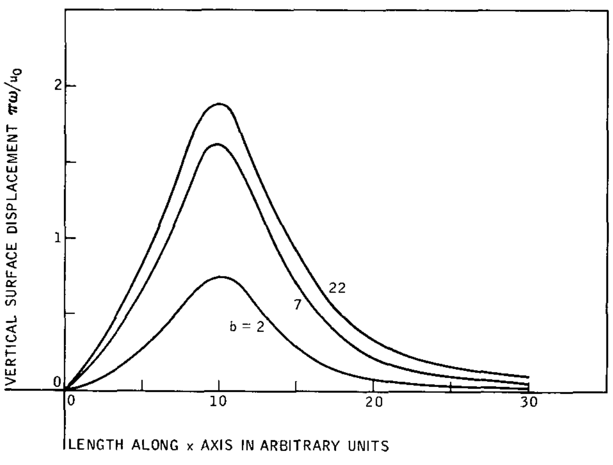
The double bar quantities are obtained by writing $z_i + h_i$ for $z_i - h_i$ in the single bar quantities.

Figures 1 and 2 show the vertical component of the displacement at the free surface due to a constant displacement discontinuity (or dislocation) in the x-component over a rectangular area inside the elastic half-space. The calculations are based upon values of elastic constants contained in a paper by Huntington.[10]

The vertical displacement at the free surface is found to be anti-symmetrical with respect to the y-axis. Therefore, only the quantities to the right of the y-axis are plotted.

Figure 1 shows plots for the vertical displacement component along the x-axis for three different shapes of the rectangular area and with the elastic properties of the half-space taken to be those of barium titanate ceramic.

Figure 1 **Surface vertical displacement due to dislocation u_0 over a rectangular area at depth h in barium titanate ceramic.** *Rectangle is centered on origin of x—y axes with length 2b in y-direction and length 2a in x-direction. Here, a = 10 units and depth h = 5 units.*



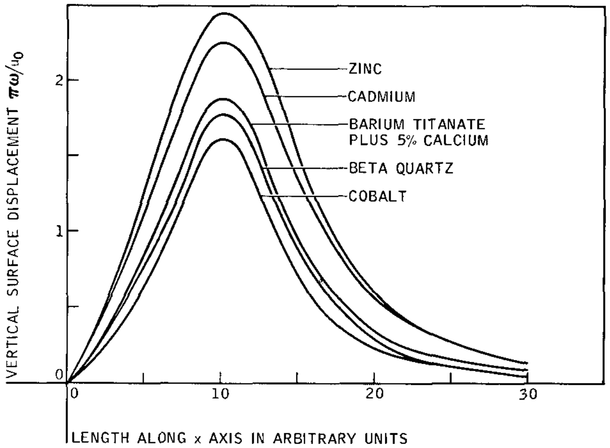


Figure 2 **Vertical displacement at surface due to dislocation u_0 over a rectangular area at depth h.** *Rectangle is located as in Fig. 1 with a = 10 units, b = 12 units, and h = 5 units.*

In Fig. 2 the vertical displacement along the x-axis is plotted for different elastic materials. In both cases, the depth of the rectangular area from the free surface is the same.

Appendix

If the area of the displacement discontinuity is a circle, it is particularly convenient to use cylindrical co-ordinates. Consider a set of cylindrical co-ordinates (r, θ, z). Then, at the plane $z = 0$, we have the following boundary conditions, corresponding to Eqs. (8) to (13):

$$\sigma_{zz} = \sigma'_{zz} \qquad u_\theta + \bar{u}_\theta = u'_\theta$$
$$\sigma_{rz} = \sigma'_{rz} \qquad u_r + \bar{u}_r = u'_r$$
$$\sigma_{\theta z} = \sigma'_{\theta z} \qquad w = w', \quad (A.1)$$

where the functions $\bar{u}_\theta$ and $\bar{u}_r$ represent the desired discontinuity over a circle A of radius a and are defined to be zero everywhere outside this circle, over the $z = 0$ plane. Again, let

$$\phi_i(r, \theta, z_i) = -\phi_i'(r, \theta, -z_i)$$

$$\psi(r, \theta, z_3) = -\psi'(r, \theta, -z_3) \quad (A.2)$$

$$\phi_1 = \frac{(k_2 + 1)}{(k_2 - k_1)} H(r, \theta, z_1) \quad (A.3)$$

$$\phi_2 = -\frac{(k_1 + 1)}{(k_2 - k_1)} H(r, \theta, z_2) \quad (A.4)$$

and $\nabla^2 H(r, \theta, z) = 0$.

$H(r, \theta, z)$ and $\psi(r, \theta, z)$ are now expressed in terms of integrals involving the Bessel function $J_n(r)$ as follows:

$$H(r, \theta, z) = \sum_{n=-\infty}^{\infty} e^{in\theta} \int_0^\infty h_n(\xi)\ J_n(r\xi)\, e^{-\xi z}\, d\xi$$

$$\psi(r, \theta, z) = \sum_{n=-\infty}^{\infty} -ie^{in\theta} \int_0^\infty g_n(\xi)\ J_n(r\xi)\, e^{-\xi z}\, d\xi. \quad (A.5)$$

Then at $z = 0$

$$-\tfrac{1}{2}\bar{u}_r = \sum_{n=-\infty}^{\infty} e^{in\theta} \int_0^\infty \left[h_n(\xi)\, \xi J_n'(r\xi) + \frac{n}{r} g_n(\xi)\ J_n(r\xi) \right] d\xi$$

$$-\tfrac{1}{2}\bar{u}_\theta = \sum_{n=-\infty}^{\infty} ie^{in\theta} \int_0^\infty \left[\frac{n}{r} h_n(\xi)\ J_n(r\xi) + \xi g_n(\xi)\ J_n'(r\xi) \right] d\xi. \quad (A.6)$$

Assume that $\bar{u}_r$ and $\bar{u}_\theta$ can be expanded in the form

$$-\tfrac{1}{2}\bar{u}_r = \sum_{n=-\infty}^{\infty} e^{in\theta} \bar{u}_{rn}(r)$$

$$-\tfrac{1}{2}\bar{u}_\theta = i \sum_{n=-\infty}^{\infty} e^{in\theta} \bar{u}_{\theta n}(r). \quad (A.7)$$

Also

$$h_n(\xi) = \alpha_n(\xi) + \beta_n(\xi) \text{ and } g_n(\xi) = \alpha_n(\xi) - \beta_n(\xi). \quad (A.8)$$

Then

$$\bar{u}_{rn} = \int_0^\infty \alpha_n(\xi)\, \xi J_{n-1}(r\xi)\, d\xi - \int_0^\infty \beta_n(\xi)\, \xi J_{n+1}(r\xi)\, d\xi$$

$$\bar{u}_{\theta n} = \int_0^\infty \alpha_n(\xi)\, \xi J_{n-1}(r\xi)\, d\xi + \int_0^\infty \beta_n(\xi)\, \xi J_{n+1}(r\xi)\, d\xi.$$

Therefore

$$\alpha_n(\xi) = \frac{1}{2} \int_0^a J_{n-1}(r\xi)(\bar{u}_{rn} + \bar{u}_{\theta n}) r\, dr$$

$$\beta_n(\xi) = -\frac{1}{2} \int_0^a J_{n+1}(r\xi)(\bar{u}_{rn} - \bar{u}_{\theta n}) r\, dr. \quad (A.9)$$

This solves the general problem for a circular discontinuity.

Example 1

If at $z = 0$, $\bar{u}_r = 0$ and $\bar{u}_\theta = -2Ar$, then from Eqs. (A.7), (A.9)

$$-\alpha_0(\xi) = \beta_0(\xi) = -A \int_0^a J_1(r\xi) r^2\, dr = -\frac{Aa^2}{\xi} J_2(\xi a).$$

Example 2

If at $z = 0$, $\bar{u}_y = 0$ and $\bar{u}_x = 1$, then one finds that

$$\alpha_1(\xi) = -\beta_{-1}(\xi) = \frac{a}{2\xi} J_1(a\xi)$$

and

$$H(r, \theta, z) = \tfrac{1}{2} \cos\theta \int_0^\infty \frac{a}{\xi} J_1(a\xi)\, J_1(r\xi) e^{-\xi z}\, d\xi$$

$$\psi(r, \theta, z) = \tfrac{1}{2} \sin\theta \int_0^\infty \frac{a}{\xi} J_1(a\xi)\, J_1(r\xi) e^{-\xi z}\, d\xi.$$

In both of the examples stresses and displacements will involve integrals of the type

$$I(\mu, \nu, \lambda) = \int_0^\infty e^{-pt} t^\lambda J_\mu(at)\ J_\nu(bt)\, dt.$$

For discussion of properties of this type of integral including elliptic integral representations see Luke.[11] Eason, Noble, and Sneddon[12] have given power series representations and recurrence formulae of them, and have also composed tables for some values of (μ, ν, λ).

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